

Towards the Fradkin-Vasiliev formalism in three dimensions

Yu. M. Zinoviev *

*Institute for High Energy Physics
of National Research Center "Kurchatov Institute"
Protvino, Moscow Region, 142280, Russia*

Abstract

In this paper we show that using frame-like gauge invariant formulation for the massive bosonic and fermionic fields in three dimensions the free Lagrangians for these fields can be rewritten in the explicitly gauge invariant form in terms of the appropriately chosen set of gauge invariant objects. This in turn opens the possibility to apply the Fradkin-Vasiliev formalism to the investigation of possible interactions of such fields.

*E-mail address: Yuri.Zinoviev@ihep.ru

Introduction

One of the effective ways to investigate interactions for higher spin fields in the Fradkin-Vasiliev formalism [1,2] (see also [3,4]). Initially this formalism was developed for the massless higher spin fields (some examples may be found in [5–9]). But the most important ingredients of such formalism are frame-like formalism and gauge invariance and the frame-like gauge invariant description exists for the massive higher spins as well [10,11]. Thus, in-principle, this formalism can be applied to the investigation of possible interactions for any system with massive and/or (partially) massless fields (some examples can be found in [12–15]).

At the same time it is a common belief that the Fradkin-Vasiliev formalism operates in dimensions equal or greater than four only. Indeed, as far as the massless higher spins in $d = 3$ are concerned, one has mostly deals with the Chern-Simons theories (e.g. [16–19]) that is tightly connected with the fact that such massless fields do not have any local physical degrees of freedom being pure gauges. But there are three cases when higher spin fields in $d = 3$ do have some physical degrees of freedom: bosonic massive field, bosonic partially massless field of the maximal depth and fermionic massive one. In this paper we show that for all these cases one can rewrite the free Lagrangians in the explicitly gauge invariant way in terms of the appropriate set of gauge invariant objects. This, in turn, can serve as a starting point for the application of Fradkin-Vasiliev formalism to investigation of possible interactions for such fields. Our construction will be based on the frame-like gauge invariant formulation for the massive bosonic and fermionic higher spin fields in $d = 3$ [20,21] (see also [22–24]).

The paper is organized as follows. In Section 1 we give a short review of the main features of the Fradkin-Vasiliev formalism. We separately consider massless and massive case and show that in the massive case there is a possibility to extend the formalism to three dimensions. In Section 2 we consider bosonic higher spin field (both massive as well as partially massless cases) while in Section 3 we consider massive fermionic case. In the Appendix using massive spin 2 as the simplest non-trivial example we show that it is indeed possible to adopt the results of [11] to three dimensions. But the analogous result for the arbitrary spin would require a lot of rather complicated calculations so in the main part of the paper we find all necessary formulas from scratch directly in $d = 3$.

Notations and conventions. We use a frame-like multispinor formalism where all objects (one-forms or zero-forms) have local indices which are completely symmetric spinor ones. To simplify expressions we will use condensed notations for the spinor indices such that e.g.

$$\Omega^{\alpha(2k)} = \Omega^{(\alpha_1 \alpha_2 \dots \alpha_{2k})}$$

Also we will always assume that spinor indices denoted by the same letter and placed on the same level are symmetrized, e.g.

$$\Omega^{\alpha(2k)} \zeta^\alpha = \Omega^{(\alpha_1 \dots \alpha_{2k}} \zeta^{\alpha_{2k+1})}$$

AdS_3 space will be described by the background frame (one-form) $e^{\alpha(2)}$ and the covariant derivative D normalized so that

$$D \wedge D \zeta^\alpha = -\lambda^2 E^\alpha{}_\beta \zeta^\beta$$

where two-form $E^{\alpha(2)}$ is defined as follows:

$$e^{\alpha(2)} \wedge e^{\beta(2)} = \varepsilon^{\alpha\beta} E^{\alpha\beta}$$

In the most part of the paper the wedge product sign \wedge will be omitted.

1 Fradkin-Vasiliev formalism

In this section we provide brief review of the Fradkin-Vasiliev formalism. Our aim here to look for the possibilities to apply this formalism to higher spins in three dimensions. We begin with the massless case and then we consider the massive one.

1.1 Massless case

In the frame-like formalism the description of the massless higher spin requires a collection of one-forms (physical, auxiliary and extra ones) with some local indices which we collectively denote as Φ . Each field is a gauge field with its own gauge transformation (schematically):

$$\delta\Phi \sim D\xi \oplus e\xi$$

For each field (physical, auxiliary or extra one) we can construct a gauge invariant two-form (curvature):

$$\mathcal{R} \sim D \wedge \Phi \oplus e \wedge \Phi$$

The free Lagrangian can then be rewritten in terms of these curvatures in the explicitly gauge invariant form:

$$\mathcal{L}_0 = \sum c_k \mathcal{R}_k \wedge \mathcal{R}_k$$

where coefficients are fixed by the so called extra field decoupling condition. As it well known for the massless fields such description requires that the cosmological constant be non zero. At the same time from the structure of the Lagrangian (the sum of the squares of two-forms) it is clear that the space-time dimension must be greater or equal to four. Thus it is not possible to apply such formalism to the massless higher spins in $d = 3$.

Let us turn to the construction of cubic vertices in this formalism. The first step is to construct the most general quadratic deformations for all these gauge invariant curvatures:

$$\mathcal{R} \Rightarrow \hat{\mathcal{R}} = \mathcal{R} \oplus \Phi \wedge \Phi$$

Here the most important requirement is that these deformed curvatures transform covariantly under the all gauge symmetries:

$$\delta\hat{\mathcal{R}} \sim \mathcal{R}\xi$$

Then one consider the following ansatz for the interacting Lagrangian:

$$\mathcal{L} \sim \sum \hat{\mathcal{R}} \wedge \hat{\mathcal{R}} \oplus \sum \mathcal{R} \wedge \mathcal{R} \wedge \Phi$$

Here the first part is just the free Lagrangian but with the curvatures replaced by the deformed ones, while the second part contains all possible abelian (or Chern-Simons like) vertices. In the linear approximation the variations of this Lagrangian lead to the expressions quadratic in the curvatures and the requirement for the Lagrangian be gauge invariant is reduced to a set of simple algebraic equations on the coefficients. It has been shown [3] that all non-trivial cubic vertices for the massless higher spins can be reproduced in such a way.

1.2 Massive case

As it can be seen from the massless case two most important ingredients of the Fradkin-Vasiliev formalism are frame-like formalism and gauge invariance. But the frame-like gauge invariant description exists for the massive higher spins as well [10, 11] and it opens the possibility to apply this formalism to the construction of the cubic vertices containing massive and/or massless higher spins. Let us stress the main differences between massless and massive cases.

The frame-like gauge invariant formulations necessarily requires not only a set of one-forms Φ but also a set of zero-forms C which transform non-trivially under the gauge transformations:

$$\delta\Phi \sim D\xi + \dots, \quad \delta C \sim \xi$$

Thus the zero-forms play the role of the Stueckelberg fields and their appearance is quite natural because in the massive case we have to expect that all gauge symmetries must be spontaneously broken.

Similarly to the massless case, for each field one can also construct a gauge invariant object (two-form for the one-form field and one-form for the zero-form one):

$$\begin{aligned} \mathcal{R} &\sim D\Phi \oplus e\Phi \oplus eeC \\ \mathcal{B} &\sim DC \oplus \Phi \oplus eC \end{aligned}$$

Using these gauge invariant objects one can rewrite the free Lagrangian in the explicitly gauge invariant form:

$$\mathcal{L}_0 \sim \mathcal{R}\mathcal{R} \oplus \mathcal{R}\mathcal{B} \oplus \mathcal{B}\mathcal{B}$$

The coefficients in these expression also must satisfy the extra field decoupling condition, but in the massive case their solution is not unique. The reason is that some combinations of these quadratic terms form total derivative. All such combinations can be systematically generated using differential identities on these gauge invariant objects. Now if we managed to find a solution where all the terms of the type $\mathcal{R}\mathcal{R}$ are absent we will obtain the form of the free Lagrangian that is valid in three dimensions as well. In the Appendix, using the massive spin-2 case as the simplest non-trivial example, we show that it is indeed possible. But for the arbitrary spin case it would require a lot of complicated calculations. So in this paper we will work directly in three dimensions and we will obtain all necessary formulas from scratch. Note also, that contrary to the massless case here the cosmological constant need not be non-zero so that such formalism works in the flat Minkowski space as well.

Let us turn to the construction of cubic vertices. Here we also begin with the most general quadratic deformations for all gauge invariant objects:

$$\begin{aligned} \hat{\mathcal{R}} &\sim \mathcal{R} \oplus \Phi\Phi \oplus e\Phi C \oplus eeCC \\ \hat{\mathcal{B}} &\sim \mathcal{B} \oplus \Phi C \oplus eCC \end{aligned}$$

and require that they transform covariantly:

$$\delta\hat{\mathcal{R}} \sim \mathcal{R}\xi, \quad \delta\hat{\mathcal{B}} \sim \mathcal{B}\xi$$

Due to the presence of the zero-forms there exists a lot of possible field re-definitions that one has take into account:

$$\begin{aligned}\Phi &\Rightarrow \Phi \oplus \Phi C \oplus eCC \\ C &\Rightarrow C \oplus CC\end{aligned}$$

Then the interacting Lagrangian can be constructed as the free Lagrangian where all gauge invariant objects are replaced by the deformed one plus all possible abelian vertices.

2 Integer spin

In this section we consider massive bosonic field with integer spin $s \geq 2$.

2.1 General massive case

The gauge invariant description for massive bosonic spin- s field [20] uses a collection of massless fields with spins $s, s-1, \dots, 0$. In the frame-like approach we need pairs of one-forms $(\Omega^{\alpha(2k)}, \Phi^{\alpha(2k)})$, $1 \leq k \leq s-1$, a zero-form $B^{\alpha(2)}$ and one-form A for the spin-1 component as well as two zero-forms $(\pi^{\alpha(2)}, \varphi)$ for the spin-0 one. The Lagrangian describing massive spin- s field in the $(A)dS_3$ background has the form:

$$\begin{aligned}\mathcal{L}_0 &= \sum_{k=1}^{s-1} (-1)^{k+1} [k \Omega_{\alpha(2k-1)\beta} e^\beta{}_\gamma \Omega^{\alpha(2k-1)\gamma} + \Omega_{\alpha(2k)} D \Phi^{\alpha(2k)}] \\ &\quad + E B_{\alpha\beta} B^{\alpha\beta} - B_{\alpha\beta} e^{\alpha\beta} D A - E \pi_{\alpha\beta} \pi^{\alpha\beta} + \pi_{\alpha\beta} E^{\alpha\beta} D \varphi \\ &\quad + \sum_{k=1}^{s-2} (-1)^{k+1} b_k \left[-\frac{(k+2)}{k} \Omega_{\alpha(2)\beta(2k)} e^{\alpha(2)} \Phi^{\beta(2k)} + \Omega_{\alpha(2k)} e_{\beta(2)} \Phi^{\alpha(2k)\beta(2)} \right] \\ &\quad + 2b_0 \Omega_{\alpha(2)} e^{\alpha(2)} A - b_0 \Phi_{\alpha\beta} E^\beta{}_\gamma B^{\alpha\gamma} + 8c_1 \pi_{\alpha\beta} E^{\alpha\beta} A \\ &\quad + \sum_{k=1}^{s-1} (-1)^{k+1} c_k \Phi_{\alpha(2k-1)\beta} e^\beta{}_\gamma \Phi^{\alpha(2k-1)\gamma} + \frac{M s b_0}{2} \Phi_{\alpha(2)} E^{\alpha(2)} \varphi + \frac{3b_0^2}{2} E \varphi^2\end{aligned}\quad (1)$$

where

$$\begin{aligned}b(k)^2 &= \frac{k(s+k+1)(s-k-1)}{2(k+1)(k+2)(2k+3)} [m^2 - (s+k)(s-k-2)\Lambda] \\ b_0^2 &= \frac{(s+1)(s-1)}{3} [m^2 - s(s-2)\Lambda] \\ c_k &= \frac{s^2 M^2}{4k(k+1)^2}, \quad M^2 = m^2 - (s-1)^2 \Lambda\end{aligned}$$

Note that the structure of this Lagrangian corresponds to the general pattern for the gauge invariant description of massive higher spin fields. Namely, the first two lines and the last one are just the sum of kinetic and mass-like terms for all components, while the third and the fourth lines contain cross-terms gluing all these components together. The most important property of this approach (that completely determines the very structure and all the coefficients) is that it allows one to keep all the gauge symmetries that massless

components initially have. Indeed, this Lagrangian is invariant under the following set of gauge transformations:

$$\begin{aligned}
\delta\Omega^{\alpha(2k)} &= D\eta^{\alpha(2k)} + \frac{(k+2)b_k}{k}e_{\beta(2)}\eta^{\alpha(2k)\beta(2)} \\
&\quad + \frac{b_{k-1}}{k(2k-1)}e^{\alpha(2)}\eta^{\alpha(2k-2)} + \frac{c_k}{k}e^\alpha_\beta\xi^{\alpha(2k-1)\beta} \\
\delta\Phi^{\alpha(2k)} &= D\xi^{\alpha(2k)} + e^\alpha_\beta\eta^{\alpha(2k-1)\beta} + b_k e_{\beta(2)}\xi^{\alpha(2k)\beta(2)} \\
&\quad + \frac{(k+1)b_{k-1}}{k(k-1)(2k-1)}e^{\alpha(2)}\xi^{\alpha(2k-2)} \\
\delta\Omega^{\alpha(2)} &= D\eta^{\alpha(2)} + 3b_1 e_{\beta(2)}\eta^{\alpha(2)\beta(2)} + c_1 e^\alpha_\gamma \xi^{\alpha\gamma} \\
\delta\Phi^{\alpha(2)} &= D\xi^{\alpha(2)} + e^\alpha_\gamma \eta^{\alpha\gamma} + b_1 e_{\beta(2)}\xi^{\alpha(2)\beta(2)} + 2b_0 e^{\alpha(2)}\xi \\
\delta B^{\alpha(2)} &= 2b_0 \eta^{\alpha(2)}, \quad \delta A = D\xi + \frac{b_0}{4}e_{\alpha(2)}\xi^{\alpha(2)} \\
\delta\pi^{\alpha(2)} &= \frac{Msb_0}{2}\xi^{\alpha(2)}, \quad \delta\varphi = -2Ms\xi
\end{aligned} \tag{2}$$

As is now very well known, in the de Sitter space ($\Lambda > 0$) there exist a number of special values for the mass m when one of the parameters $b(k_0) = 0$. In this case the whole Lagrangian decomposes into two independent subsystems one of which describes a so-called partially massless spin- s field. But most of these partially massless fields in $d = 3$ do not have any physical degrees of freedom (similarly to the massless ones). Apart from the general massive field having two physical degrees of freedom, only so-called partially massless field of the maximal depth has one degrees of freedom. This corresponds to the decoupling of the spin-0 component which happens when

$$M^2 = m^2 - (s-1)^2\Lambda = 0$$

2.2 Partially massless case of maximal depth

In this case all $c(k) = 0$ so that all the explicit mass-like terms vanish and the Lagrangian greatly simplifies:

$$\begin{aligned}
\mathcal{L}_0 &= \sum_{k=1}^{s-1} (-1)^{k+1} [k\Omega_{\alpha(2k-1)\beta} e^\beta_\gamma \Omega^{\alpha(2k-1)\gamma} + \Omega_{\alpha(2k)} D\Phi^{\alpha(2k)}] \\
&\quad + \sum_{k=1}^{s-2} (-1)^{k+1} b_k \left[-\frac{(k+2)}{k} \Omega_{\alpha(2)\beta(2k)} e^{\alpha(2)} \Phi^{\beta(2k)} + \Omega_{\alpha(2k)} e_{\beta(2)} \Phi^{\alpha(2k)\beta(2)} \right] \\
&\quad + EB_{\alpha\beta} B^{\alpha\beta} - B_{\alpha\beta} e^{\alpha\beta} DA + 2b_0 \Omega_{\alpha(2)} e^{\alpha(2)} A - b_0 f_{\alpha\beta} E^\beta_\gamma B^{\alpha\gamma}
\end{aligned} \tag{3}$$

where now:

$$\begin{aligned}
b_k^2 &= \frac{k(k+1)(s+k+1)(s-k-1)}{2(k+2)(2k+3)}\Lambda \\
b_0^2 &= \frac{(s+1)(s-1)}{3}\Lambda
\end{aligned}$$

The Lagrangian is still invariant under all the gauge transformations:

$$\begin{aligned}
\delta_0 \Omega^{\alpha(2k)} &= D\eta^{\alpha(2k)} + \frac{(k+2)b_k}{k} e_{\beta(2)} \eta^{\alpha(2k)\beta(2)} + \frac{b_{k-1}}{k(2k-1)} e^{\alpha(2)} \eta^{\alpha(2k-2)} \\
\delta_0 \Phi^{\alpha(2k)} &= D\xi^{\alpha(2k)} + e^\alpha{}_\beta \eta^{\alpha(2k-1)\beta} + b_k e_{\beta(2)} \xi^{\alpha(2k)\beta(2)} + \frac{(k+1)b_{k-1}}{k(k-1)(2k-1)} e^{\alpha(2)} \xi^{\alpha(2k-2)} \\
\delta \Omega^{\alpha(2)} &= D\eta^{\alpha(2)} + 3b_1 e_{\beta(2)} \eta^{\alpha(2)\beta(2)} \\
\delta \Phi^{\alpha(2)} &= D\xi^{\alpha(2)} + e^\alpha{}_\beta \eta^{\alpha\beta} + b_1 e_{\beta(2)} \xi^{\alpha(2)\beta(2)} + 2b_0 e^{\alpha(2)} \xi \\
\delta B^{\alpha\beta} &= 2b_0 \eta^{\alpha\beta}, \quad \delta A = D\xi + \frac{b_0}{4} e_{\alpha\beta} \xi^{\alpha\beta}
\end{aligned} \tag{4}$$

Now having in our disposal the explicit form of all gauge transformations it is not hard to construct a set of gauge invariant objects for all the fields (two-forms for the one-form fields and one-forms for the zero-form ones):

$$\begin{aligned}
\mathcal{R}^{\alpha(2k)} &= D\Omega^{\alpha(2k)} + \frac{(k+2)b_k}{k} e_{\beta(2)} \Omega^{\alpha(2k)\beta(2)} + \frac{b_{k-1}}{k(2k-1)} e^{\alpha(2)} \Omega^{\alpha(2k-2)} \\
\mathcal{T}^{\alpha(2k)} &= D\Phi^{\alpha(2k)} + e^\alpha{}_\beta \Omega^{\alpha(2k-1)\beta} + b_k e_{\beta(2)} \Phi^{\alpha(2k)\beta(2)} + \frac{(k+1)b_{k-1}}{k(k-1)(2k-1)} e^{\alpha(2)} \Phi^{\alpha(2k-2)} \\
\mathcal{R}^{\alpha(2)} &= D\Omega^{\alpha(2)} + 3b_1 e_{\beta(2)} \Omega^{\alpha(2)\beta(2)} - \frac{b_0}{2} E^\alpha{}_\beta B^{\alpha\beta} \\
\mathcal{T}^{\alpha(2)} &= D\Phi^{\alpha(2)} + e^\alpha{}_\beta \Omega^{\alpha\beta} + b_1 e_{\beta(2)} \Phi^{\alpha(2)\beta(2)} + 2b_0 e^{\alpha(2)} A \\
\mathcal{A} &= DA - E_{\alpha(2)} B^{\alpha(2)} + \frac{b_0}{4} e_{\alpha(2)} \Phi^{\alpha(2)} \\
\mathcal{B}^{\alpha(2)} &= DB^{\alpha(2)} - 2b_0 \Omega^{\alpha(2)} + 3b_1 e_{\beta(2)} B^{\alpha(2)\beta(2)}
\end{aligned} \tag{5}$$

As it can be seen from the last line, to achieve gauge invariance we introduced an extra zero-form $B^{\alpha(4)}$ playing the role of the Stueckelberg field for the $\eta^{\alpha(4)}$ transformations:

$$\delta B^{\alpha(4)} = 2b_0 \eta^{\alpha(4)}$$

But then we have to construct a gauge invariant object for this new field and this in turn requires introduction of the next extra field and so on until we exhaust all the gauge symmetries. This results in the following collection of extra zero-forms $B^{\alpha(2k)}$, $2 \leq k \leq s-1$

$$\delta B^{\alpha(2k)} = 2b_0 \eta^{\alpha(2k)} \tag{6}$$

with the corresponding set of gauge invariant one-forms:

$$\mathcal{B}^{\alpha(2k)} = DB^{\alpha(2k)} - 2b_0 \Omega^{\alpha(2k)} + \frac{(k+2)b_k}{k} e_{\beta(2)} B^{\alpha(2k)\beta(2)} + \frac{b_{k-1}}{k(2k-1)} e^{\alpha(2)} B^{\alpha(2k-2)} \tag{7}$$

In what follows we will need the differential identities for the two-forms which follows directly from their explicit expressions:

$$\begin{aligned}
D\mathcal{T}^{\alpha(2k)} &= -e^\alpha{}_\beta \mathcal{R}^{\alpha(2k-1)\beta} - b_1 e_{\beta(2)} \mathcal{T}^{\alpha(2k)\beta(2)} - \frac{(k+1)b_{k-1}}{k(k-1)(2k-1)} e^{\alpha(2)} \mathcal{T}^{\alpha(2k-2)} \\
D\mathcal{T}^{\alpha(2)} &= -e^\alpha{}_\beta \mathcal{R}^{\alpha\beta} - b_1 e_{\beta(2)} \mathcal{T}^{\alpha(2)\beta(2)} - 2b_0 e^{\alpha(2)} \mathcal{A}
\end{aligned} \tag{8}$$

as well as the similar identities for the one-forms:

$$\begin{aligned} D\mathcal{B}^{\alpha(2k)} &= -2b_0\mathcal{R}^{\alpha(2k)} - \frac{(k+2)b_k}{k}e_{\beta(2)}\mathcal{B}^{\alpha(2k)\beta(2)} - \frac{b_{k-1}}{k(2k-1)}e^{\alpha(2)}\mathcal{B}^{\alpha(2k-2)} \\ D\mathcal{B}^{\alpha(2)} &= -2b_0\mathcal{R}^{\alpha(2)} - 3b_1e_{\beta(2)}\mathcal{B}^{\alpha(2)\beta(2)} \end{aligned} \quad (9)$$

Now we have equal numbers of the two-forms $\mathcal{R}^{\alpha(2k)}$ and one-forms $\mathcal{B}^{\alpha(2k)}$, $1 \leq k \leq s-1$ and we will try to rewrite the Lagrangian in terms of these objects. For this let us consider the following ansatz:

$$\mathcal{L} = \sum_{k=1}^{s-1} (-1)^{k+1} [e_k \mathcal{T}_{\alpha(2k)} \mathcal{B}^{\alpha(2k)} + f_k \mathcal{B}_{\alpha(2k-1)\beta} e^\beta{}_\gamma \mathcal{B}^{\alpha(2k-1)\gamma}] \quad (10)$$

Clearly, by construction this Lagrangian is gauge invariant for any values of the coefficients e_k and f_k . But to reproduce our initial Lagrangian we have to adjust these coefficients so that all the extra fields decouple. Let us extract all the terms containing $B^{\alpha(2k)}$:

$$\begin{aligned} (-1)^{k+1} \Delta \mathcal{L} &= e_k \mathcal{T}_{\alpha(2k)} D\mathcal{B}^{\alpha(2k)} - e_{k+1} b_k \mathcal{T}_{\alpha(2k)\beta(2)} e^{\beta(2)} B^{\alpha(2k)} \\ &\quad - \frac{(k+1)e_{k-1}b_{k-1}}{(k-1)} \mathcal{T}_{\alpha(2k-2)} e_{\beta(2)} B^{\alpha(2k-2)\beta(2)} \\ &\quad - 2f_k D\mathcal{B}^{\alpha(2k-1)\beta} e_{\beta}{}^\gamma B_{\alpha(2k-1)\gamma} - \frac{4(k+2)f_{k+1}b_k}{(k+1)} \mathcal{B}_{\alpha(2k)\beta(2)} E^{\beta(2)} B^{\alpha(2k)} \\ &\quad - \frac{4(k+1)f_{k-1}b_{k-1}}{(k-1)} \mathcal{B}_{\alpha(2k-2)} E_{\beta(2)} B^{\alpha(2k-2)\beta(2)} \end{aligned}$$

Now using the differential identities given above we obtain:

$$\begin{aligned} -e_k D\mathcal{T}^{\alpha(2k)} B_{\alpha(2k)} &= -2ke_k \mathcal{R}_{\alpha(2k-1)\beta} e^\beta{}_\gamma B^{\alpha(2k-1)\gamma} + e_k b_k \mathcal{T}_{\alpha(2k)\beta(2)} e^{\beta(2)} B^{\alpha(2k)} \\ &\quad + \frac{(k+1)e_k b_{k-1}}{(k-1)} \mathcal{T}_{\alpha(2k-2)} e_{\beta(2)} B^{\alpha(2k-2)\beta(2)} \\ -2f_k D\mathcal{B}^{\alpha(2k-1)\beta} e_{\beta}{}^\gamma B_{\alpha(2k-1)\gamma} &= 4b_0 f_k \mathcal{R}^{\alpha(2k-1)\beta} e_{\beta}{}^\gamma B_{\alpha(2k-1)\gamma} \\ &\quad + \frac{4(k+2)f_k b_k}{k} \mathcal{B}^{\alpha(2k)\beta(2)} E_{\beta(2)} B_{\alpha(2k)} \\ &\quad + \frac{4(k+1)f_k b_{k-1}}{k} \mathcal{B}^{\alpha(2k-2)} E^{\beta(2)} B_{\alpha(2k-2)\beta(2)} \end{aligned}$$

Collecting all pieces together we finally get:

$$\begin{aligned} (-1)^{k+1} \Delta \mathcal{L} &= (4b_0 f_k - 2ke_k) \mathcal{R}_{\alpha(2k-1)\beta} e^\beta{}_\gamma B^{\alpha(2k-1)\gamma} \\ &\quad + (e_k - e_{k+1}) b_k \mathcal{T}_{\alpha(2k)\beta(2)} e^{\beta(2)} B^{\alpha(2k)} \\ &\quad + \frac{(k+1)b_{k-1}}{(k+1)} (e_k - e_{k-1}) \mathcal{T}_{\alpha(2k-2)} e_{\beta(2)} B^{\alpha(2k-2)\beta(2)} \\ &\quad + 4(k+2)b_k \left(\frac{f_k}{k} - \frac{f_{k+1}}{(k+1)} \right) \mathcal{B}_{\alpha(2k)\beta(2)} e^{\beta(2)} B^{\alpha(2k)} \\ &\quad + 4(k+1)b_{k-1} \left(\frac{f_k}{k} - \frac{f_{k-1}}{(k-1)} \right) \mathcal{B}_{\alpha(2k-2)} e_{\beta(2)} B^{\alpha(2k-2)\beta(2)} \end{aligned} \quad (11)$$

Note that to have a correct normalization of the kinetic terms for all fields we have to put

$$e_k = -\frac{1}{2b_0}, \quad f_k = -\frac{k}{4b_0^2} \quad (12)$$

and it is easy to see that in this case all the terms containing $B^{\alpha(2k)}$ above vanish.

To complete, we have to consider terms with the $B^{\alpha(2)}$ field. We get:

$$\begin{aligned} \Delta\mathcal{L} = & e_1\mathcal{T}_{\alpha(2)}DB^{\alpha(2)} - b_1e_2\mathcal{T}_{\alpha(2)\beta(2)}e^{\beta(2)}B^{\alpha(2)} \\ & + 2f_1D\mathcal{B}^{\alpha\beta}e_{\beta}{}^{\gamma}B_{\alpha\gamma} - 6b_1f_2\mathcal{B}_{\alpha(2)\beta(2)}E^{\beta(2)}B^{\alpha(2)} \end{aligned}$$

Once again using the differential identities:

$$\begin{aligned} -e_1D\mathcal{T}^{\alpha(2)}B_{\alpha(2)} &= e_1[e^{\alpha}{}_{\beta}\mathcal{R}^{\alpha\beta} + b_1e_{\beta(2)}\mathcal{T}^{\alpha(2)\beta(2)} + 2b_0e^{\alpha(2)}\mathcal{A}]B_{\alpha(2)} \\ 2f_1D\mathcal{B}^{\alpha\beta}e_{\beta}{}^{\gamma}B_{\alpha\gamma} &= -4f_1b_0\mathcal{R}^{\alpha\beta}e_{\beta}{}^{\gamma}B_{\alpha\gamma} + 12b_1f_1\mathcal{B}^{\alpha(2)\beta(2)}E_{\beta(2)}B_{\alpha(2)} \end{aligned}$$

we finally obtain:

$$\begin{aligned} \Delta\mathcal{L} = & (2e_1 - 4f_1b_0)\mathcal{R}^{\alpha\beta}e_{\beta}{}^{\gamma}B_{\alpha\gamma} + b_1(e_1 - e_2)\mathcal{T}_{\alpha(2)\beta(2)}e^{\beta(2)}B^{\alpha(2)} \\ & + 6b_1(2f_1 - f_2)\mathcal{B}_{\alpha(2)\beta(2)}e^{\beta(2)}B^{\alpha(2)} + 2b_0e_1e^{\alpha(2)}B_{\alpha(2)}\mathcal{A} \\ = & 2b_0e_1e^{\alpha(2)}B_{\alpha(2)}\mathcal{A} \end{aligned} \quad (13)$$

Thus for the chosen coefficients e_k and f_k we achieved complete decoupling for all the extra fields and the resulting Lagrangian (as we have explicitly checked) reproduces our initial one. Note at last, that this Lagrangian can be rewritten also in the more common to three dimensions CS-like form:

$$\mathcal{L} = \sum_{k=1}^{s-1} (-1)^{k+1} [\mathcal{T}_{\alpha(2k)}\Omega^{\alpha(2k)} - k\Omega_{\alpha(2k-1)\beta}e^{\beta}{}_{\gamma}\Omega^{\alpha(2k-1)\gamma}] - e^{\alpha(2)}B_{\alpha(2)}\mathcal{A} \quad (14)$$

2.3 Partial gauge fixing

In the frame-like formalism for the massless spin- s field $(\Omega^{\alpha(2s-2)}, \Phi^{\alpha(2s-2)})$ in anti-de Sitter space ($\Lambda = -\lambda^2 < 0$) there exists a very convenient possibility of the separation of variables. Namely, let us consider the Lagrangian for such massless field:

$$\begin{aligned} \mathcal{L}_0 = & (-1)^s [(s-1)\Omega_{\alpha(2s-3)\beta}e^{\beta}{}_{\gamma}\Omega^{\alpha(2s-3)\gamma} + \Omega_{\alpha(2s-2)}D\Phi^{\alpha(2s-2)} \\ & + \frac{(s-1)\lambda^2}{4}\Phi_{\alpha(2s-3)\beta}e^{\beta}{}_{\gamma}\Phi^{\alpha(2s-3)\gamma}] \end{aligned} \quad (15)$$

This Lagrangian is invariant under the following local gauge transformations:

$$\begin{aligned} \delta\Omega^{\alpha(2s-2)} &= D\eta^{\alpha(2s-2)} + \frac{\lambda^2}{4}e^{\alpha}{}_{\beta}\xi^{\alpha(2s-3)\beta} \\ \delta\Phi^{\alpha(2s-2)} &= D\xi^{\alpha(2s-2)} + e^{\alpha}{}_{\beta}\eta^{\alpha(2s-3)\beta} \end{aligned} \quad (16)$$

Let us introduce new variables:

$$\begin{aligned}\hat{\Omega}^{\alpha(2s-2)} &= \Omega^{\alpha(2s-2)} + \frac{\lambda}{2}\Phi^{\alpha(2s-2)} \\ \hat{\Phi}^{\alpha(2s-2)} &= \Omega^{\alpha(2s-2)} - \frac{\lambda}{2}\Phi^{\alpha(2s-2)}\end{aligned}\quad (17)$$

and similarly for the parameters of the gauge transformations:

$$\begin{aligned}\hat{\eta}^{\alpha(2s-2)} &= \eta^{\alpha(2s-2)} + \frac{\lambda}{2}\xi^{\alpha(2s-2)} \\ \hat{\xi}^{\alpha(2s-2)} &= \eta^{\alpha(2s-2)} - \frac{\lambda}{2}\xi^{\alpha(2s-2)}\end{aligned}\quad (18)$$

Then the Lagrangian can be rewritten as the sum of the two independent parts:

$$\begin{aligned}\mathcal{L}_0 &= \frac{(-1)^s}{2\lambda}[(s-1)\lambda\hat{\Omega}_{\alpha(2s-3)\beta}e^\beta{}_\gamma\hat{\Omega}^{\alpha(2s-3)\gamma} + \hat{\Omega}_{\alpha(2s-2)}D\hat{\Omega}^{\alpha(2s-2)} \\ &\quad + (s-1)\lambda\hat{\Phi}_{\alpha(2s-3)\beta}e^\beta{}_\gamma\hat{\Phi}^{\alpha(2s-3)\gamma} - \hat{\Phi}_{\alpha(2s-2)}D\hat{\Phi}^{\alpha(2s-2)}]\end{aligned}\quad (19)$$

while the gauge transformations take the form:

$$\begin{aligned}\delta\hat{\Omega}^{\alpha(2s-2)} &= D\hat{\eta}^{\alpha(2s-2)} + \frac{\lambda}{2}e^\alpha{}_\beta\hat{\eta}^{\alpha(2s-3)\beta} \\ \delta\hat{\Phi}^{\alpha(2s-2)} &= D\hat{\xi}^{\alpha(2s-2)} - \frac{\lambda}{2}e^\alpha{}_\beta\hat{\xi}^{\alpha(2s-3)\beta}\end{aligned}\quad (20)$$

Moreover, such separation of variables works not only in the free case but in the interacting case as well.

As we have shown previously [20], similar mechanism is possible for the massive fields provided one uses a partial gauge fixing. Namely, let us return to the general massive case, described above, and use the ξ gauge transformation to set the gauge $\varphi = 0$. Then, solving the spin-0 equation

$$A = \frac{1}{2Ms}e_{\alpha(2)}\pi^{\alpha(2)}$$

we obtain the following Lagrangian (after rescaling $\pi \Rightarrow 2Ms\pi$):

$$\begin{aligned}\mathcal{L} &= \sum_{k=1}^{s-1}(-1)^{k+1}[k\Omega_{\alpha(2k-1)\beta}e^\beta{}_\gamma\Omega^{\alpha(2k-1)\gamma} + \Omega_{\alpha(2k)}D\Phi^{\alpha(2k)} + c_k\Phi_{\alpha(2k-1)\beta}e^\beta{}_\gamma\Phi^{\alpha(2k-1)\gamma}] \\ &\quad + \sum_{k=1}^{s-2}(-1)^{k+1}b_k[-\frac{(k+2)}{k}\Omega_{\alpha(2)\beta(2k)}e^{\alpha(2)}\Phi^{\beta(2k)} + \Omega_{\alpha(2k)}e_{\beta(2)}\Phi^{\alpha(2k)\beta(2)}] \\ &\quad + EB_{\alpha\beta}B^{\alpha\beta} + 4B_{\alpha\gamma}E^{\alpha\beta}D\pi_\beta{}^\gamma + 4M^2s^2E\pi_{\alpha\beta}\pi^{\alpha\beta} \\ &\quad + 8b_0\Omega_{\alpha\gamma}E^{\alpha\beta}\pi_\beta{}^\gamma - b_0\Phi_{\alpha\beta}E^\beta{}_\gamma B^{\alpha\gamma}\end{aligned}\quad (21)$$

Now let us introduce new variables:

$$\begin{aligned}\hat{\Omega}^{\alpha(2k)} &= \Omega^{\alpha(2k)} + \frac{Ms}{2k(k+1)}\Phi^{\alpha(2k)} \\ \hat{\Phi}^{\alpha(2k)} &= \Omega^{\alpha(2k)} - \frac{Ms}{2k(k+1)}\Phi^{\alpha(2k)}\end{aligned}\quad (22)$$

and similarly for the zero-forms:

$$\begin{aligned}\hat{B}^{\alpha(2)} &= B^{\alpha(2)} + 2Ms\pi^{\alpha(2)} \\ \hat{\pi}^{\alpha(2)} &= B^{\alpha(2)} - 2Ms\pi^{\alpha(2)}\end{aligned}\tag{23}$$

then the Lagrangian also decomposes into two independent parts, one for the fields $(\hat{\Omega}, \hat{B})$ and the other one for the fields $(\hat{\Phi}, \hat{\pi})$. From now on, we consider the first part only. The Lagrangian has the form (omitting hats):

$$\begin{aligned}\mathcal{L} &= \sum_{k=1}^{s-1} (-1)^{k+1} \left[\frac{k}{2} Ms \Omega_{\alpha(2k-1)\beta} e^{\beta}{}_{\gamma} \Omega^{\alpha(2k-1)\gamma} + \frac{k(k+1)}{2} \Omega_{\alpha(2k)} D\Omega^{\alpha(2k)} \right] \\ &\quad - \sum_{k=1}^{s-2} (-1)^{k+1} (k+1)(k+2) b_k \Omega_{\alpha(2k)\beta(2)} e^{\beta(2)} \Omega^{\alpha(2k)} \\ &\quad + \frac{Ms}{2} E B_{\alpha(2)} B^{\alpha(2)} + \frac{1}{2} B_{\alpha\gamma} E^{\alpha\beta} D B_{\beta}{}^{\gamma} + 2b_0 \Omega_{\alpha\gamma} E^{\alpha\beta} B_{\beta}{}^{\gamma}\end{aligned}\tag{24}$$

This Lagrangian is invariant under its own half of the gauge transformations:

$$\begin{aligned}\delta\Omega^{\alpha(2k)} &= D\eta^{\alpha(2k)} + \frac{Ms}{2k(k+1)} e^{\alpha}{}_{\beta} \eta^{\alpha(2k-1)\beta} \\ &\quad + \frac{(k+2)}{k} b_k e_{\beta(2)} \eta^{\alpha(2k)\beta(2)} + \frac{b_{k-1}}{k(2k-1)} e^{\alpha(2)} \eta^{\alpha(2k-2)} \\ \delta\Omega^{\alpha(2)} &= D\eta^{\alpha(2)} + \frac{Ms}{4} e^{\alpha}{}_{\beta} \eta^{\alpha\beta} + 3b_1 e_{\beta(2)} \eta^{\alpha(2)\beta(2)} \\ \delta B^{\alpha(2)} &= 2b_0 \eta^{\alpha(2)}\end{aligned}\tag{25}$$

Now we proceed with the construction of the gauge invariant objects for all the fields:

$$\begin{aligned}\mathcal{R}^{\alpha(2k)} &= D\Omega^{\alpha(2k)} + \frac{Ms}{2k(k+1)} e^{\alpha}{}_{\beta} \Omega^{\alpha(2k-1)\beta} + \frac{(k+2)}{k} b_k e_{\beta(2)} \Omega^{\alpha(2k)\beta(2)} \\ &\quad + \frac{b_{k-1}}{k(2k-1)} e^{\alpha(2)} \Omega^{\alpha(2k-2)} \\ \mathcal{R}^{\alpha(2)} &= D\Omega^{\alpha(2)} + \frac{Ms}{4} e^{\alpha}{}_{\beta} \Omega^{\alpha\beta} + 3b_1 e_{\beta(2)} \Omega^{\alpha(2)\beta(2)} - \frac{b_0}{2} E^{\alpha}{}_{\beta} B^{\alpha\beta} \\ \mathcal{B}^{\alpha(2)} &= DB^{\alpha(2)} - 2b_0 \Omega^{\alpha(2)} + \frac{Ms}{4} e^{\alpha}{}_{\beta} B^{\alpha\beta} + 3b_1 e_{\beta(2)} B^{\alpha(2)\beta(2)}\end{aligned}\tag{26}$$

Again, to achieve the gauge invariance for the $\mathcal{B}^{\alpha(2)}$ we introduced an extra field $B^{\alpha(4)}$:

$$\delta B^{\alpha(4)} = 2b_0 \eta^{\alpha(4)}$$

As in the previous case, the procedure ends up with the set of such extra zero-forms $B^{\alpha(2k)}$, $2 \leq k \leq s-1$

$$\delta B^{\alpha(2k)} = 2b_0 \eta^{\alpha(2k)}\tag{27}$$

with the appropriate set of gauge invariant one-forms:

$$\begin{aligned}\mathcal{B}^{\alpha(2k)} &= DB^{\alpha(2k)} - 2b_0\Omega^{\alpha(2k)} + \frac{Ms}{2k(k+1)}e^\alpha{}_\beta B^{\alpha(2k-1)\beta} \\ &\quad + \frac{b_{k-1}}{k(2k-1)}e^{\alpha(2)} B^{\alpha(2k-2)} + \frac{(k+2)}{k}b_k e_{\beta(2)} B^{\alpha(2k)\beta(2)}\end{aligned}\quad (28)$$

In what follows we will need the differential identities for the two-forms only:

$$\begin{aligned}D\mathcal{R}^{\alpha(2k)} &= -\frac{Ms}{2k(k+1)}e^\alpha{}_\beta \mathcal{R}^{\alpha(2k-1)\beta} - \frac{(k+2)}{k}b_k e_{\beta(2)} \mathcal{R}^{\alpha(2k)\beta(2)} \\ &\quad - \frac{b_{k-1}}{k(2k-1)}e^{\alpha(2)} \mathcal{R}^{\alpha(2k-2)} \\ D\mathcal{R}^{\alpha(2)} &= -\frac{Ms}{4}e^\alpha{}_\beta \mathcal{R}^{\alpha\beta} - 3b_1 e_{\beta(2)} \mathcal{R}^{\alpha(2)\beta(2)} - \frac{b_0}{2}E^\alpha{}_\beta \mathcal{B}^{\alpha\beta}\end{aligned}\quad (29)$$

Now let us try to rewrite the Lagrangian in the explicitly gauge invariant form. It happens that in this case it is enough to consider the following ansatz:

$$\mathcal{L} = \sum (-1)^k e_k \mathcal{R}_{\alpha(2k)} \mathcal{B}^{\alpha(2k)} \quad (30)$$

Indeed, let us extract all the terms with the extra field $B^{\alpha(2k)}$:

$$\begin{aligned}(01)^{k+1} \Delta \mathcal{L} &= e_k [\mathcal{R}_{\alpha(2k)} DB^{\alpha(2k)} + \frac{Ms}{(k+1)} \mathcal{R}_{\alpha(2k-1)\beta} e^\beta{}_\gamma B^{\alpha(2k-1)\gamma}] \\ &\quad - e_{k+1} b_k \mathcal{R}_{\alpha(2k)\beta(2)} e^{\beta(2)} B^{\alpha(2k)} \\ &\quad - \frac{(k+1)}{(k-1)} b_{k-1} e_{k-1} \mathcal{R}_{\alpha(2k-2)} e_{\beta(2)} B^{\alpha(2k-2)\beta(2)}\end{aligned}$$

One more time using the differential identity:

$$\begin{aligned}-e_k D\mathcal{R}^{\alpha(2k)} B_{\alpha(2k)} &= -\frac{Ms}{(k+1)} e_k \mathcal{R}_{\alpha(2k-1)\beta} e^\beta{}_\gamma B^{\alpha(2k-1)\gamma} \\ &\quad + \frac{(k+2)}{k} e_k b_k \mathcal{R}_{\alpha(2k)\beta(2)} e^{\beta(2)} B^{\alpha(2k)} \\ &\quad + e_k b_{k-1} \mathcal{R}_{\alpha(2k-2)} e_{\beta(2)} B^{\alpha(2k-2)\beta(2)}\end{aligned}$$

we obtain finally:

$$\begin{aligned}(-1)^{k+1} \Delta \mathcal{L} &= \left(\frac{(k+2)}{k} e_k - e_{k+1}\right) b_k \mathcal{R}_{\alpha(2k)\beta(2)} e^{\beta(2)} B^{\alpha(2k)} \\ &\quad + \left(e_k - \frac{(k+1)}{(k-1)} e_{k-1}\right) b_{k-1} \mathcal{R}_{\alpha(2k-2)} e_{\beta(2)} B^{\alpha(2k-2)\beta(2)}\end{aligned}\quad (31)$$

In this case to have the correct normalization for the kinetic terms we have to put:

$$e_k = -\frac{k(k+1)}{4b_0} \quad \Rightarrow \quad e_{k-1} = \frac{(k-1)}{(k+1)} e_k \quad (32)$$

and it is easy to see that all the terms with the $B^{\alpha(2k)}$ fields vanish.

To complete, we have to consider terms with the $B^{\alpha(20)}$ field as well. We get:

$$\Delta\mathcal{L} = -e_1 D\mathcal{R}_{\alpha(2)} B^{\alpha(2)} + \frac{Mse_1}{2} \mathcal{R}_{\alpha\beta} e^\beta{}_\gamma B^{\alpha\gamma} - b_1 e_2 \mathcal{R}_{\alpha(2)\beta(2)} e^{\beta(2)} B^{\alpha(2)}$$

while the differential identity leads to

$$-e_1 D\mathcal{R}^{\alpha(2)} B_{\alpha(2)} = -\frac{Mse_1}{4} \mathcal{R}_{\alpha\beta} e^\beta{}_\gamma B_{\alpha\gamma} + 3b_1 e_1 \mathcal{R}_{\alpha(2)\beta(2)} e^{\beta(2)} B^{\alpha(2)} - b_0 e_1 E^\alpha{}_\beta \mathcal{B}^{\beta\gamma} B_{\alpha\gamma}$$

Thus we obtain:

$$\begin{aligned} \Delta\mathcal{L} &= (3e_1 - e_2) b_1 \mathcal{R}_{\alpha(2)\beta(2)} e^{\beta(2)} B^{\alpha(2)} - b_0 e_1 E^\alpha{}_\beta \mathcal{B}^{\beta\gamma} B_{\alpha\gamma} \\ &= -b_0 e_1 E^\alpha{}_\beta \mathcal{B}^{\beta\gamma} B_{\alpha\gamma} \end{aligned} \quad (33)$$

So our ansatz with the coefficients e_k given above does reproduce our initial Lagrangian (as we have explicitly checked). Due to the decoupling of all the extra fields this Lagrangian can be also rewritten in the CS-like form:

$$\mathcal{L} = \sum_{k=1}^{s-1} (-1)^{k+1} \frac{k(k+1)}{2} \mathcal{R}_{\alpha(2k)} \Omega^{\alpha(2k)} + \frac{1}{2} E^\alpha{}_\beta \mathcal{B}^{\beta\gamma} B_{\alpha\gamma} \quad (34)$$

3 Half-integer spin

In this section we consider a massive fermionic field with half-integer spin $s+1/2$. The frame-like gauge invariant formulation [21] requires a set of one-forms $\Phi^{\alpha(2k+1)}$, $0 \leq k \leq s-1$, as well as zero-form ϕ^α . The Lagrangian describing such massive field in $(A)dS_3$ background has the form:

$$\begin{aligned} \frac{1}{i} \mathcal{L} &= \sum_{k=0}^{s-1} (-1)^{k+1} \left[\frac{1}{2} \Phi_{\alpha(2k+1)} D\Phi^{\alpha(2k+1)} \right] + \frac{1}{2} \phi_\alpha E^\alpha{}_\beta D\phi^\beta \\ &\quad + \sum_{k=1}^{s-1} (-1)^{k+1} a_k \Phi_{\alpha(2k-1)\beta(2)} e^{\beta(2)} \Phi^{\alpha(2k-1)} + a_0 \Phi_\alpha E^\alpha{}_\beta \phi^\beta \\ &\quad + \sum_{k=0}^{s-1} (-1)^{k+1} \frac{b_k}{2} \Phi_{\alpha(2k)\beta} e^\beta{}_\gamma \Phi^{\alpha(2k)\gamma} - \frac{3b_0}{2} E\phi_\alpha \phi^\alpha \end{aligned} \quad (35)$$

where

$$\begin{aligned} b_k &= \frac{(2s+1)}{(2k+3)} M, \quad M^2 = m^2 - (s - \frac{1}{2})^2 \Lambda \\ a_k^2 &= \frac{(s+k+1)(s-k)}{2(k+1)(2k+1)} [M^2 + (2k+1)^2 \frac{\Lambda}{4}] \\ a_0^2 &= 2s(s+1) [M^2 + \frac{\Lambda}{4}] \end{aligned}$$

The structure of this Lagrangian repeats the general pattern: the first and the third lines are the sum of the kinetic and mass-like terms for all the components, while the second line contains all necessary cross-terms.

This Lagrangian is invariant under the following gauge transformations with the fermionic parameters:

$$\begin{aligned}\delta\Phi^{\alpha(2k+1)} &= D\xi^{\alpha(2k+1)} + \frac{b_k}{(2k+1)}e^\alpha{}_\beta\xi^{\alpha(2k)\beta} \\ &\quad + \frac{a_k}{k(2k+1)}e^{\alpha(2)}\xi^{\alpha(2k-1)} + a_{k+1}e_{\beta(2)}\xi^{\alpha(2k+1)\beta(2)} \\ \delta\phi^\alpha &= a_0\xi^\alpha\end{aligned}\tag{36}$$

Now we construct a set of gauge invariant objects for all fields (two-forms for Φ and one-form for ϕ):

$$\begin{aligned}\mathcal{R}^{\alpha(2k+1)} &= D\Phi^{\alpha(2k+1)} + \frac{b_k}{(2k+1)}e^\alpha{}_\beta\Phi^{\alpha(2k)\beta} + \frac{a_k}{k(2k+1)}e^{\alpha(2)}\Phi^{\alpha(2k-1)} \\ &\quad + a_{k+1}e_{\beta(2)}\Phi^{\alpha(2k+1)\beta(2)} \\ \mathcal{R}^\alpha &= D\Phi^\alpha + b_0e^\alpha{}_\beta\Phi^\beta + a_1e_{\beta(2)}\Phi^{\alpha\beta(2)} - a_0E^\alpha{}_\beta\phi^\beta \\ \mathcal{F}^\alpha &= D\phi^\alpha - a_0\Phi^\alpha + b_0e^\alpha{}_\beta\phi^\beta + a_1e_{\beta(2)}\phi^{\alpha\beta(2)}\end{aligned}\tag{37}$$

Similarly to the bosonic case, to achieve gauge invariance for the last one we introduced an extra zero-form $\phi^{\alpha(3)}$ playing the role of the Stueckelberg field for the $\xi^{\alpha(3)}$ transformations:

$$\delta\phi^{\alpha(3)} = a_0\xi^{\alpha(3)}$$

The whole procedure ends with the set of such extra zero-forms $\phi^{\alpha(2k+1)}$, $1 \leq k \leq s-1$ with the appropriate gauge invariant one-forms:

$$\begin{aligned}\mathcal{F}^{\alpha(2k+1)} &= D\phi^{\alpha(2k+1)} - a_0\Phi^{\alpha(2k+1)} + \frac{b_k}{(2k+1)}e^\alpha{}_\beta\phi^{\alpha(2k)\beta} \\ &\quad + \frac{a_k}{k(2k+1)}e^{\alpha(2)}\phi^{\alpha(2k-1)} + a_{k+1}e_{\beta(2)}\phi^{\alpha(2k+1)\beta(2)}\end{aligned}\tag{38}$$

where

$$\delta\phi^{\alpha(2k+1)} = a_0\xi^{\alpha(2k+1)}\tag{39}$$

In what follows we will need the differential identities for the two-forms:

$$\begin{aligned}D\mathcal{R}^{\alpha(2k+1)} &= -\frac{b_k}{(2k+1)}e^\alpha{}_\beta\mathcal{R}^{\alpha(2k)\beta} - \frac{a_k}{k(2k+1)}e^{\alpha(2)}\mathcal{R}^{\alpha(2k-1)} \\ &\quad - a_{k+1}e_{\beta(2)}\mathcal{R}^{\alpha(2k+1)\beta(2)} \\ D\mathcal{R}^\alpha &= -b_0e^\alpha{}_\beta\mathcal{R}^\beta - a_1e_{\beta(2)}\mathcal{R}^{\alpha\beta(2)} - a_0E^\alpha{}_\beta\mathcal{F}^\beta\end{aligned}\tag{40}$$

Now having in our disposal an equal number of two-forms and one-forms we will try to rewrite the Lagrangian in the explicitly gauge invariant form. For this we consider the following ansatz:

$$\mathcal{L} = \sum_{k=0}^{s-1} (-1)^{k+1} e_k \mathcal{R}_{\alpha(2k+1)} \mathcal{T}^{\alpha(2k+1)}\tag{41}$$

We have to adjust the coefficients e_k in such a way so that all extra zero-forms decouple. Let us extract all the terms containing the zero-form $\phi^{\alpha(2k+1)}$:

$$\begin{aligned} (-1)^{k+1} \Delta \mathcal{L} &= e_k D \mathcal{R}^{\alpha(2k+1)} \phi_{\alpha(2k+1)} + e_k b_k \mathcal{R}_{\alpha(2k)\beta} e^\beta_\gamma \phi^{\alpha(2k)\gamma} \\ &\quad - e_{k+1} a_{k+1} \mathcal{R}_{\alpha(2k+1)\beta(2)} e^{\beta(2)} \phi^{\alpha(2k+1)} \\ &\quad - e_{k-1} a_k \mathcal{R}_{\alpha(2k-1)} e_{\beta(2)} \phi^{\alpha(2k-1)\beta(2)} \end{aligned}$$

With the help of the differential identity for $\mathcal{R}^{\alpha(2k+1)}$ that gives:

$$\begin{aligned} e_k D \mathcal{R}^{\alpha(2k+1)} \phi_{\alpha(2k+1)} &= -e_k b_k \mathcal{R}_{\alpha(2k)\beta} e^\beta_\gamma \phi^{\alpha(2k)\gamma} \\ &\quad + e_k a_{k+1} \mathcal{R}_{\alpha(2k+1)\beta(2)} e^{\beta(2)} \phi^{\alpha(2k+1)} \\ &\quad + e_k a_k \mathcal{R}_{\alpha(2k-1)} e_{\beta(2)} \phi^{\alpha(2k-1)\beta(2)} \end{aligned}$$

we obtain finally:

$$\begin{aligned} \Delta \mathcal{L} &= (e_k - e_{k+1}) a_{k+1} \mathcal{R}_{\alpha(2k+1)\beta(2)} e^{\beta(2)} \phi^{\alpha(2k+1)} \\ &\quad + (e_k - e_{k-1}) a_k \mathcal{R}_{\alpha(2k-1)} e_{\beta(2)} \phi^{\alpha(2k-1)\beta(2)} \end{aligned} \quad (42)$$

In this case to get the correct normalization for the kinetic terms we have to put:

$$e_k = -\frac{i}{2a_0} \quad (43)$$

and as a result all the terms containing extra fields vanish.

To complete we have to consider the terms with ϕ^α :

$$\Delta \mathcal{L} = e_0 D \mathcal{R}^\alpha \phi_\alpha + e_0 b_0 \mathcal{R}_\alpha e^\alpha_\beta \phi^\beta - e_1 a_1 \mathcal{R}_{\alpha\beta(2)} e^{\beta(2)} \phi^\alpha$$

Once again using the differential identity

$$e_0 D \mathcal{R}^\alpha \phi_\alpha = -e_0 b_0 \mathcal{R}_\alpha e^\alpha_\beta \phi^\beta + e_0 a_1 \mathcal{R}_{\alpha\beta(2)} e^{\beta(2)} \phi^\alpha + e_0 a_0 \mathcal{F}_\alpha E^\alpha_\beta \phi^\beta$$

we obtain:

$$\begin{aligned} \Delta \mathcal{L} &= (e_0 - e_1) a_1 \mathcal{R}_{\alpha\beta(2)} e^{\beta(2)} \phi^\alpha + e_0 a_0 \mathcal{F}_\alpha E^\alpha_\beta \phi^\beta \\ &= e_0 a_0 \mathcal{F}_\alpha E^\alpha_\beta \phi^\beta \end{aligned} \quad (44)$$

Thus we obtained the explicitly gauge invariant expression that correctly reproduce our Lagrangian (as we have checked). As the last remark note that this Lagrangian also can be rewritten in the CS-like form:

$$\frac{1}{i} \mathcal{L} = \sum_{k=0}^{s-1} (-1)^{k+1} \frac{1}{2} \mathcal{R}_{\alpha(2k+1)} \Phi^{\alpha(2k+1)} - \frac{1}{2} \mathcal{F}_\alpha E^\alpha_\beta \phi^\beta \quad (45)$$

Conclusion

Thus we have shown that using the frame-like gauge invariant formalism for the massive (and partially massless) higher spin fields in three dimensions the free Lagrangian for these fields can be rewritten in the explicitly gauge invariant form in terms of the appropriately chosen set of gauge invariant objects. These gauge invariant objects necessarily contain not only Lagrangian fields but a set of extra fields as well and these extra fields play an important role in the interacting theories. At the same time the free Lagrangians are constructed in such a way that all these extra fields decouple.

An open question is how to deal with the systems containing both massless and massive higher spins because the massless fields are naturally described by the Chern-Simons theories. Note however that in both cases (massless and massive) we have a set of gauge invariant objects. So we certainly can proceed with the first stage of FV-formalism, i.e. with the deformation procedure. But the construction of the Lagrangian formalism for the interacting theories is still has to be developed.

Acknowledgments

Author is grateful to I. L. Buchbinder and T. V. Snegirev for collaboration. The author acknowledges a kind hospitality extended to him at the MIAPP program "Higher Spin Theory and Duality" (Munich, May 2-27 2016) where this talk was given. Work was supported in parts by RFBR grant No. 14-02-01172.

A Massive spin-2 in $d \geq 3$

In this appendix, using the massive spin-2 case as an example, we illustrate the relation of our three-dimensional results with their higher-dimensional analogs.

The frame-like gauge invariant formulation for the massive spin-2 field [10] requires three pairs of auxiliary and physical fields: $(\Omega_\mu^{ab}, f_\mu^a)$, (B^{ab}, B_μ) and (π^a, φ) . The Lagrangian describing such massive field in $(A)dS_d$ background ($d \geq 3$) has the form:

$$\begin{aligned} \mathcal{L}_0 = & \frac{1}{2} \{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \} \Omega_\mu^{ac} \Omega_\nu^{bc} - \frac{1}{2} \{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \} \Omega_\mu^{ab} D_\nu f_\alpha^c + \frac{1}{2} B_{ab}^2 \\ & - \{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \} B^{ab} D_\mu B_\nu - \frac{(d-2)}{2(d-1)} \pi_a^2 + \frac{(d-2)}{(d-1)} e^\mu{}_a \pi^a D_\mu \varphi \\ & + m [\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \} \Omega_\mu^{ab} B_\nu + e^\mu{}_a B^{ab} f_\mu^b] - 2M e^\mu{}_a \pi^a B_\mu \\ & + \frac{M^2}{2} \{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \} f_\mu^a f_\nu^b - m M e^\mu{}_a f_\mu^a \varphi + \frac{d}{2(d-1)} m^2 \varphi^2 \end{aligned} \quad (46)$$

where

$$M^2 = m^2 - \kappa(d-2) \quad (47)$$

This Lagrangian is invariant under the following gauge transformations:

$$\delta \Omega_\mu^{ab} = D_\mu \eta^{ab} - \frac{M^2}{(d-2)} e_\mu^{[a} \xi^{b]}$$

$$\begin{aligned}
\delta f_\mu^a &= D_\mu \xi^a + \eta_\mu^a + \frac{2m}{(d-2)} e_\mu^a \xi \\
\delta B_\mu &= D_\mu \xi + \frac{m}{2} \xi_\mu, \quad \delta B^{ab} = -m \eta^{ab} \\
\delta \varphi &= \frac{2(d-1)}{(d-2)} M \xi, \quad \delta \pi^a = -\frac{(d-1)}{(d-2)} m M \xi^a
\end{aligned} \tag{48}$$

For all the six fields we can construct the gauge invariant objects (two-forms or one-forms):

$$\begin{aligned}
\mathcal{F}_{\mu\nu}^{ab} &= D_{[\mu} \Omega_{\nu]}^{ab} - \frac{m}{(d-2)} e_{[\mu}^{[a} B_{\nu]}^{b]} - \frac{M^2}{(d-2)} e_{[\mu}^{[a} f_{\nu]}^{b]} + \frac{2mM}{(d-1)(d-2)} e_{[\mu}^a e_{\nu]}^b \varphi \\
T_{\mu\nu}^a &= D_{[\mu} f_{\nu]}^a - \Omega_{[\mu,\nu]}^a + \frac{2m}{(d-2)} e_{[\mu}^a B_{\nu]} \\
\mathcal{B}_\mu^{ab} &= D_\mu B^{ab} + m \Omega_\mu^{ab} - \frac{M}{(d-1)} e_\mu^{[a} \pi^{b]} \\
\mathcal{B}_{\mu\nu} &= D_{[\mu} B_{\nu]} - B_{\mu\nu} - \frac{m}{2} f_{[\mu,\nu]} \\
\Pi_\mu^a &= D_\mu \pi^a + \frac{(d-1)}{(d-2)} M B_\mu^a + \frac{(d-1)}{(d-2)} m M f_\mu^a - \frac{m^2}{(d-2)} e_\mu^a \varphi \\
\Phi_\mu &= D_\mu \varphi - \pi_\mu - \frac{2(d-1)}{(d-2)} M B_\mu
\end{aligned} \tag{49}$$

They satisfy the following differential identities:

$$\begin{aligned}
D_{[\mu} \mathcal{F}_{\nu\alpha]}^{ab} &= \frac{m}{(d-2)} e_{[\mu}^{[a} \mathcal{B}_{\nu,\alpha]}^{b]} + \frac{M^2}{(d-2)} e_{[\mu}^{[a} T_{\nu\alpha]}^{b]} + \frac{2mM}{(d-1)(d-2)} e_{[\mu}^a e_{\nu]}^b \Phi_{\alpha]} \\
D_{[\mu} T_{\nu\alpha]}^a &= -\mathcal{F}_{[\mu\nu,\alpha]}^a - \frac{2m}{(d-2)} e_{[\mu}^a \mathcal{B}_{\nu\alpha]} \\
D_{[\mu} \mathcal{B}_{\nu]}^{ab} &= m \mathcal{F}_{\mu\nu}^{ab} + \frac{M}{(d-1)} e_{[\mu}^{[a} \Pi_{\nu]}^{b]} \\
D_{[\mu} \mathcal{B}_{\nu\alpha]} &= -\mathcal{B}_{[\mu,\nu\alpha]} - \frac{m}{2} T_{[\mu\nu,\alpha]} \\
D_{[\mu} \Pi_{\nu]}^a &= \frac{(d-1)}{(d-2)} M \mathcal{B}_{[\mu,\nu]}^a + \frac{(d-1)}{(d-2)} m M T_{\mu\nu}^a + \frac{m^2}{(d-2)} e_{[\mu}^a \Phi_{\nu]} \\
D_{[\mu} \Phi_{\nu]} &= -\Pi_{[\mu,\nu]} - \frac{2(d-1)}{(d-2)} M \mathcal{B}_{\mu\nu}
\end{aligned} \tag{50}$$

Let us consider the most general ansatz for the Lagrangian in terms of these objects:

$$\begin{aligned}
\mathcal{L}_0 &= a_1 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} \mathcal{F}_{\mu\nu}^{ab} \mathcal{F}_{\alpha\beta}^{cd} + a_2 \left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} \mathcal{B}_\mu^{ac} \mathcal{B}_\nu^{bc} + a_3 \left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} \Pi_\mu^a \Pi_\nu^b \\
&\quad + a_4 \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} \mathcal{F}_{\mu\nu}^{ab} \Pi_\alpha^c + a_5 \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} T_{\mu\nu}^a \mathcal{B}_\alpha^{bc} + a_6 \left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} \mathcal{B}_\mu^{ab} \Phi_\nu
\end{aligned} \tag{51}$$

Not all these terms are independent as can be shown by considering the following identities valid up to the total derivative:

$$\left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} D_\mu [\mathcal{F}_{\nu\alpha}^{ab} \mathcal{B}_\beta^{cd}] \approx 0, \quad \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} D_\mu [\mathcal{B}_\nu^{ab} \Pi_\alpha^c] \approx 0$$

Using the differential identities given above it is rather straightforward task to obtain two relations on these six terms:

$$\begin{aligned} \frac{m}{2}X_1 + \frac{8(d-3)m}{(d-2)}X_2 + \frac{2(d-3)M}{(d-1)}X_4 + \frac{2(d-3)M^2}{(d-2)}X_5 - \frac{4(d-3)mM}{(d-1)}X_6 &= 0 \\ -\frac{2(d-1)M}{(d-2)}X_2 + \frac{2(d-2)M}{(d-1)}X_3 + \frac{m}{2}X_4 - \frac{(d-1)mM}{2(d-2)}X_5 + m^2X_6 &= 0 \end{aligned}$$

where X_i denotes term with the coefficient a_i . Thus we have a whole family of possible solutions with two free parameters. Let us provide here just two concrete examples. The first one is the choice of the authors of [11]:

$$\mathcal{L} = \frac{(d-2)}{32(d-3)M^2} \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} \mathcal{F}_{\mu\nu}{}^{ab} \mathcal{F}_{\alpha\beta}{}^{cd} + \frac{1}{2M^2} \left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} \mathcal{B}_\mu{}^{ac} \mathcal{B}_\nu{}^{bc} - \frac{(d-2)}{2(d-1)M} \left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} \mathcal{B}_\mu{}^{ab} \Phi_\nu \quad (52)$$

This solution admits the massless limit for the non-zero cosmological constant (but not the partially massless one) and is applicable for $d \geq 4$ only. The other possible solution has the form:

$$\mathcal{L} = -\frac{1}{2m^2} \left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} \mathcal{B}_\mu{}^{ac} \mathcal{B}_\nu{}^{bc} + \frac{(d-2)^2}{2(d-1)^2m^2} \left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} \Pi_\mu{}^a \Pi_\nu{}^b - \frac{1}{4m} \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} T_{\mu\nu}{}^a \mathcal{B}_\alpha{}^{bc} \quad (53)$$

This solution does not admit the massless limit (independently of the cosmological constant) but it is nicely works for $d = 3$ case. Moreover the structure of the Lagrangian is similar to the three-dimensional ones considered in this work.

References

- [1] E. S. Fradkin, M. A. Vasiliev "On the gravitational interaction of massless higher-spin fields", Phys. Lett. **B189** (1987) 89.
- [2] E. S. Fradkin, M. A. Vasiliev "Cubic interaction in extended theories of massless higher-spin fields", Nucl. Phys. **B291** (1987) 141.
- [3] M. Vasiliev "Cubic Vertices for Symmetric Higher-Spin Gauge Fields in (A)dS_d", Nucl. Phys. **B862** (2012) 341, arXiv:1108.5921.
- [4] Nicolas Boulanger, Dmitry Ponomarev, E.D. Skvortsov "Non-abelian cubic vertices for higher-spin fields in anti-de Sitter space", JHEP **1305** (2013) 008, arXiv:1211.6979.
- [5] M. A. Vasiliev "Cubic Interactions of Bosonic Higher Spin Gauge Fields in AdS₅", Nucl.Phys. **B616** (2001) 106-162; Erratum-ibid. **B652** (2003) 407, arXiv:hep-th/0106200.
- [6] K. B. Alkalaev, M. A. Vasiliev "N=1 Supersymmetric Theory of Higher Spin Gauge Fields in AdS(5) at the Cubic Level", Nucl.Phys. **B655** (2003) 57-92, arXiv:hep-th/0206068.

- [7] K.B. Alkalaev "*FV-type action for $AdS(5)$ mixed-symmetry fields*", JHEP **1103** (2011) 031, arXiv:1011.6109.
- [8] Nicolas Boulanger, E. D. Skvortsov, Yu. M. Zinoviev "*Gravitational cubic interactions for a simple mixed-symmetry gauge field in AdS and flat backgrounds*", J. Phys. **A44** (2011) 415403, arXiv:1107.1872.
- [9] Nicolas Boulanger, E. D. Skvortsov "*Higher-spin algebras and cubic interactions for simple mixed-symmetry fields in AdS spacetime*", JHEP **1109** (2011) 063, arXiv:1107.5028.
- [10] Yu. M. Zinoviev "*Frame-like gauge invariant formulation for massive high spin particles*", Nucl. Phys. **B808** (2009) 185, arXiv:0808.1778.
- [11] D. S. Ponomarev, M. A. Vasiliev "*Frame-Like Action and Unfolded Formulation for Massive Higher-Spin Fields*", Nucl. Phys. **B839** (2010) 466, arXiv:1001.0062.
- [12] Yu. M. Zinoviev "*On electromagnetic interactions for massive mixed symmetry field*", JHEP **03** (2011) 082, arXiv:1012.2706.
- [13] Yu. M. Zinoviev "*Gravitational cubic interactions for a massive mixed symmetry gauge field*", Class. Quantum Grav. **29** (2012) 015013, arXiv:1107.3222.
- [14] Yu. M. Zinoviev "*Massive spin-2 in the Fradkin-Vasiliev formalism. I. Partially massless case*", Nucl. Phys. **B886** (2014) 712, arXiv:1405.4065.
- [15] I. L. Buchbinder, T. V. Snegirev, Yu. M. Zinoviev "*Formalism of gauge invariant curvatures and constructing the cubic vertices for massive spin-3/2 field in AdS_4 space*", Eur. Phys. J. C **74** (2014) 3153, arXiv:1405.7781.
- [16] M. P. Blencowe "*A consistent interacting massless higher-spin field theory in $D=2+1$* ", Class. Quant. Grav. **6** (1989) 443.
- [17] A. Achucarro, P. K. Townsend "*A Chern-Simons Action for Three-Dimensional anti-De Sitter Supergravity Theories*", Phys. Lett. **B180** (1986) 89.
- [18] A. Campoleoni, S. Fredenhagen, S. Pfenninger, S. Theisen "*Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields*", JHEP **11** (2010) 007, arXiv:1008.4744.
- [19] Yu. M. Zinoviev "*Hypergravity in AdS_3* ", Phys. Lett. **B739** (2014) 106, arXiv:1408.2912.
- [20] I. L. Buchbinder, T. V. Snegirev, Yu. M. Zinoviev "*Gauge invariant Lagrangian formulation of massive higher spin fields in $(A)dS_3$ space*", Phys. Lett. **B716** (2012) 243-248, arXiv:1207.1215.
- [21] I. L. Buchbinder, T. V. Snegirev, Yu. M. Zinoviev "*Frame-like gauge invariant Lagrangian formulation of massive fermionic higher spin fields in AdS_3 space*", Phys. Lett. **B738** (2014) 258, arXiv:1407.3918.

- [22] I. L. Buchbinder, T. V. Snegirev, Yu. M. Zinoviev "*On gravitational interactions for massive higher spins in AdS_3* ", J. Phys. A **46** (2013) 214015, arXiv:1208.0183.
- [23] I. L. Buchbinder, T. V. Snegirev, Yu. M. Zinoviev "*Lagrangian formulation of the massive higher spin supermultiplets in three dimensional space-time*", JHEP **10** (2015) 148, arXiv:1508.02829.
- [24] I. L. Buchbinder, T. V. Snegirev, Yu. M. Zinoviev "*Unfolded equations for massive higher spin supermultiplets in AdS_3* ", arXiv:1606.02475.